

第十章

曲线积分与曲面积分

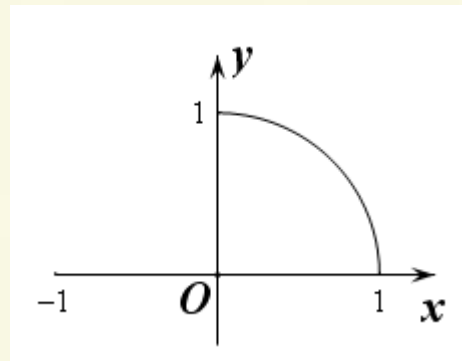
同步练习

10.1.一

1. 设 C 是曲线 $y = \sqrt{1-x^2}$ 从点 $(1,0)$ 到点 $(0,1)$ 的弧段,
则 $\int_C x \, ds = \underline{\hspace{2cm}}$

解 $C: y = \sqrt{1-x^2}$

x 为参变量, $0 \leq x \leq 1$



$$y'(x) = -\frac{x}{\sqrt{1-x^2}} \quad 1 + y'^2(x) = \frac{1}{1-x^2}$$

$$\int_C x \, ds = \int_0^1 x \sqrt{1 + y'^2(x)} \, dx = \int_0^1 \frac{x}{\sqrt{1-x^2}} \, dx$$

$$= [-\sqrt{1-x^2}]_0^1$$

$$= 1$$

10.1.一

2. 设 L 是曲线 $x = e^t \cos t, y = e^t \sin t, z = e^t$

从 $t = 0$ 到 $t = 2$ 的弧段, 则 $\int_L \frac{1}{x^2 + y^2 + z^2} ds = \underline{\hspace{2cm}}$

解 $L: t$ 为参变量, $0 \leq t \leq 2$

$$\frac{1}{x^2 + y^2 + z^2} = \frac{1}{e^{2t} \cos^2 t + e^{2t} \sin^2 t + e^{2t}} = \frac{1}{2e^{2t}}$$

$$x'(t) = e^t(\cos t - \sin t) \quad z'(t) = e^t$$

$$y'(t) = e^t(\sin t + \cos t) \quad x'^2(t) + y'^2(t) + z'^2(t) = 3e^{2t}$$

$$\int_L \frac{1}{x^2 + y^2 + z^2} ds = \int_0^2 \frac{1}{2e^{2t}} \sqrt{x'^2(t) + y'^2(t) + z'^2(t)} dt$$

$$= \int_0^2 \frac{\sqrt{3}}{2} e^{-t} dt = \left[-\frac{\sqrt{3}}{2} e^{-t} \right]_0^2$$

$$= \frac{\sqrt{3}}{2} (1 - e^{-2})$$

10.1.一

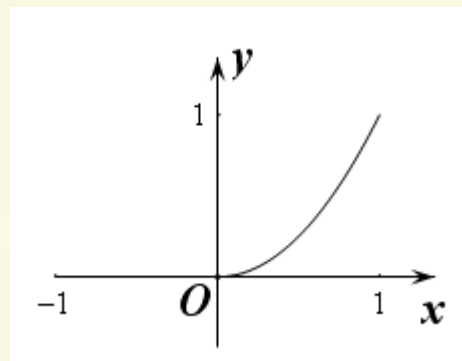
3. 设 C 是抛物线 $y = x^2$ 从点 $O(0,0)$ 到点 $A(1,1)$ 的弧段,
则 $\int_C x^2 dy = \underline{\hspace{2cm}}$

解 $C: y = x^2$ x 为参变量,

$$x_0 = 0, \quad x_A = 1,$$

$$y'(x) = 2x$$

$$\int_C x^2 dy = \int_0^1 x^2 y'(x) dx = \int_0^1 2x^3 dx = \left[\frac{1}{2} x^4 \right]_0^1 = \frac{1}{2}$$



10.1.一

4. 设 L 是曲线 $x = t, y = t^2, z = t^3$ 从 $t = 0$ 到 $t = 1$ 的弧段, 则 $\int_L (y^2 - z^2) dx + 2yzdy - x^2dz = \underline{\hspace{2cm}}$

解 L : t 为参变量, $0 \leq t \leq 1$

$$x'(t) = 1 \quad y'(t) = 2t \quad z'(t) = 3t^2$$

$$\int_L (y^2 - z^2) dx + 2yzdy - x^2dz$$

$$= \int_0^1 (t^4 - t^6) dt + 2t^2 \cdot t^3 \cdot 2t dt - t^2 \cdot 3t^2 dt$$

$$= \int_0^1 (3t^6 - 2t^4) dt$$

$$= \left[\frac{3}{7} t^7 - \frac{2}{5} t^5 \right]_0^1 = \frac{1}{35}$$

10.1.二

1. 求 $\int_C (x^3 + y^2) ds$,

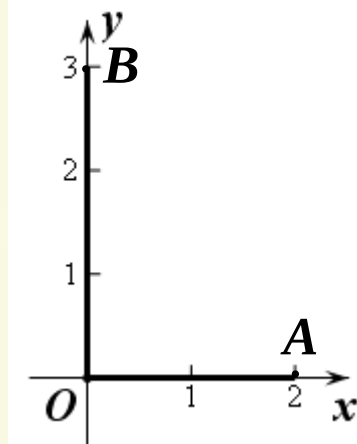
式中 C 是连接点 $A(2,0)$, $O(0,0)$ 到 $B(0,3)$ 的折线段.

解 C : $OA: y=0$ x 为参变量, $0 \leq x \leq 2$

$$y'(x) = 0 \quad 1 + y'^2(x) = 1$$

$OB: x=0$ y 为参变量, $0 \leq y \leq 3$

$$x'(y) = 0 \quad 1 + x'^2(y) = 1$$



$$\int_C (x^3 + y^2) ds = \int_{OA+OB} (x^3 + y^2) ds$$

$$= \int_0^2 (x^3 + 0) dx + \int_0^3 (0 + y^2) dy$$

$$= \int_0^2 x^3 dx + \int_0^3 y^2 dy = \left[\frac{1}{4} x^4 \right]_0^2 + \left[\frac{1}{3} y^3 \right]_0^3$$

$$= 4 + 9 = 13$$

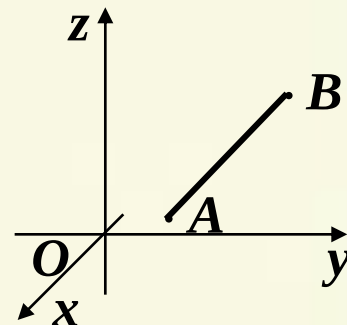
10.1.二

2. 求 $\int_L (x + y - z) ds$,

式中 L 是连接点 $A(1,2,1), B(2,4,3)$ 的直线段.

解 $L: AB$ 的参数方程为
$$\begin{cases} x = t + 1 \\ y = 2t + 2 \\ z = 2t + 1 \end{cases}$$

t 为参变量, $0 \leq t \leq 1$



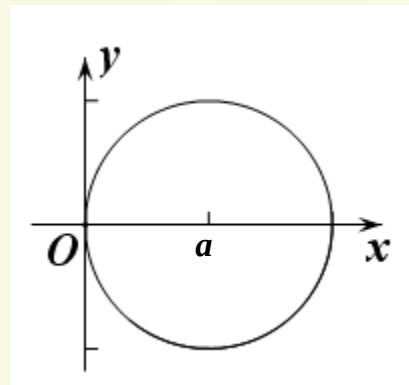
$$\begin{aligned} \int_L (x + y - z) ds &= \int_0^1 (t + 1 + 2t + 2 - 2t - 1) \sqrt{1 + 4 + 4} dt \\ &= \int_0^1 3(t + 2) dt \\ &= 3 \left[\frac{1}{2} t^2 + 2t \right]_0^1 \\ &= \frac{15}{2} \end{aligned}$$

10.1.二

3. 求 $\oint_C \sqrt{x^2 + y^2} ds$, 式中 C 是圆周 $x^2 + y^2 = ax, (a > 0)$

解 C : 极坐标方程为 $r = 2\cos\theta$

θ 为参变量, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$



$$\oint_C \sqrt{x^2 + y^2} ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2} \sqrt{r^2 + r'^2(\theta)} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2a \cos\theta \sqrt{4a^2 \cos^2\theta + 4a^2 \sin^2\theta} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2a \cos\theta \cdot 2a d\theta$$

$$= 4a^2 [\sin\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 8a^2$$

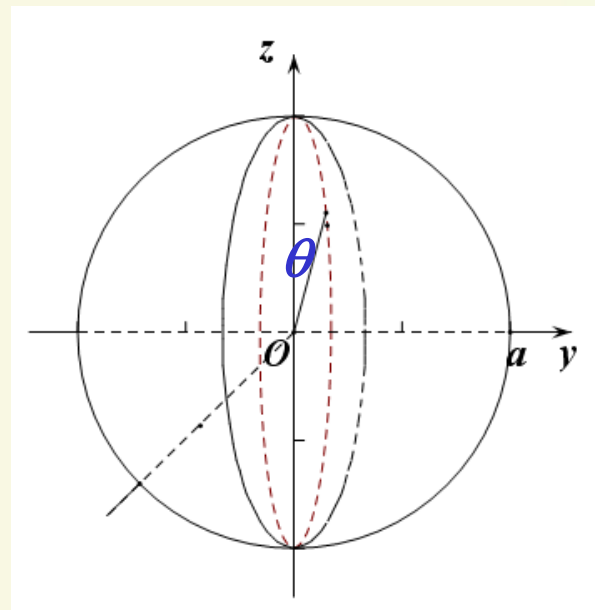
10.1.二

4. 求 $\oint_L \sqrt{2y^2 + z^2} ds$,

式中 L 是球面 $x^2 + y^2 + z^2 = a^2$ 与 $y = x$ 的交线.

解

设 $L: \begin{cases} x = \frac{a}{\sqrt{2}} \sin \theta \\ y = \frac{a}{\sqrt{2}} \sin \theta \\ z = a \cos \theta \end{cases} \quad \theta \in [0, 2\pi]$



$$I = \int_0^{2\pi} \sqrt{2\left(\frac{a}{\sqrt{2}} \sin \theta\right)^2 + (a \cos \theta)^2} \cdot \sqrt{\left(\frac{a}{\sqrt{2}} \cos \theta\right)^2 + \left(\frac{a}{\sqrt{2}} \cos \theta\right)^2 + (a \sin \theta)^2} d\theta$$

$$= \int_0^{2\pi} \sqrt{a^2} \cdot \sqrt{a^2} d\theta = \int_0^{2\pi} a \cdot a d\theta = 2\pi a^2$$

10.1.二

5. 求 $\int_C 2\sqrt{y}dx + (x^2 - y)dy$

式中 C 是抛物线 $y = x^2$ 从点 $A(1,1)$ 到点 $B(2,4)$ 的弧段.

解 $C: AB: y = x^2$ x 为参变量,

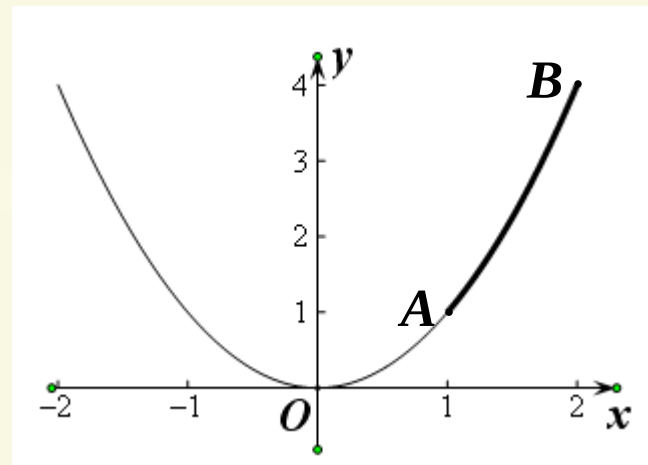
$$1 \leq x \leq 2 \quad y'(x) = 2x$$

$$\int_C 2\sqrt{y}dx + (x^2 - y)dy$$

$$= \int_1^2 2x dx + 0 \cdot 2x dx$$

$$= \int_1^2 2x dx$$

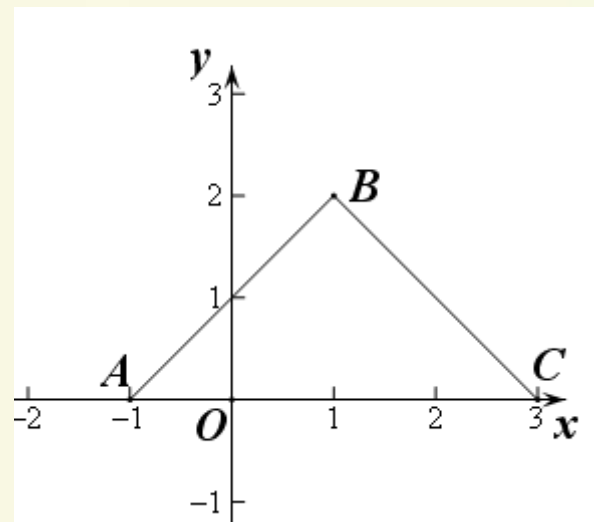
$$= [x^2]_1^2 = 3$$



10.1.二

6. 求 $\int_C (x+y-3)(y-x)dx - (x-y+1)(x+y)dy$,
式中 C 是从点 $A(-1,0)$ 沿直线 $y=x+1$ 到点 $B(1,2)$,
再沿 $x+y=3$ 的到点 $D(3,0)$ 的折线段.

$$\begin{aligned}\text{解 原式} &= \int_{AB} + \int_{BD} \\ &= \int_{-1}^1 [(2x-2) \cdot 1 - 0 \cdot (2x+1)] \cdot dx \\ &\quad + \int_1^3 [0 \cdot (3-2x) - (2x-2) \cdot 3 \cdot (-1)] dx \\ &= \int_{-1}^1 2(x-1) dx + \int_1^3 6(x-1) dx \\ &= [(x-1)^2]_{-1}^1 + 3[(x-1)^2]_1^3\end{aligned}$$



10.1.二

7. 求 $\int_C x \sin(xy) dx - y \cos(xy) dy$,
式中 C 是从点 $O(0,0)$ 到点 $A(1,\pi)$ 的直线段.

解

10.1.二

8.求 $\int_C (x^2 + y^2)dx + (x^2 - y^2)dy$,

式中 C 是曲线 $y = 1 - |1 - x|$ 上从对应 $x = 1$ 的点到 $x = 2$ 的点.

解

10.1.二

9. 求 $\int_L x^3 dx + 3zy^2 dy - x^2 y dz$,

式中 L 是从点 $A(3,2,1)$ 到点 $O(0,0,0)$ 的直线段.

解 AB 方程为
$$\begin{cases} x = 3 - 3t \\ y = 2 - 2t \\ z = 1 - t \end{cases} \quad t_A = 0, t_B = 1$$

$$\int_L x^3 dx + 3zy^2 dy - x^2 y dz$$

$$= \int_0^1 [(3-3t)^3(-3) + 3(1-t)(2-2t)^2(-2) - (3-3t)^2(2-2t)(-1)] dt$$

$$= -87 \int_0^1 (1-t)^3 dt$$

$$= \frac{87}{4} [(1-t)^4]_0^1 = -\frac{87}{4}$$

10.1.二

10. 一质点沿曲线 $x = 0, y = t, z = t^2$ 从点 $O(0, 0, 0)$ 移动到点 $A(0, 1, 1)$, 求此过程力 $F = \{\sqrt{1+x^4}, -y, 1\}$ 所做的功.

解
$$\begin{aligned} W &= \int_{OA} \sqrt{1+x^4} dx - y dy + dz \\ &= \int_0^1 (\sqrt{1+0 \cdot 0} - t \cdot 1 + 2t) dt \\ &= \int_0^1 t dt \\ &= \left[\frac{1}{2} t^2 \right]_0^1 = \frac{1}{2} \end{aligned}$$